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On the stability of double bubbles and double drops

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Abstract

The stability problems for equilibrium standard “double bubble” and “double drop” configurations are considered. Recent work has shown that a double bubble is stable to volume-preserving perturbations. In this paper, the stability to perturbations that do not conserve the volumes of the individual bubbles is examined. It is shown that a double bubble shape is also stable to these perturbations and, thus, is stable to arbitrary perturbations. The analysis is based on the principle of minimum total energy. A variational principle is used to formulate the stability problem for an equilibrium double drop configuration formed under zero gravity by two drops of immiscible incompressible liquids.
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1. Introduction

A double bubble may be formed when two soap bubbles come into contact. Equilibrium standard double bubble configurations are shown in Fig. 1. The three intersecting surfaces that comprise the configurations consist of two spherical segments that meet a third surface at equal 120° angles. The surface separating the two equidimensional bubbles (Fig. 1a) is a plane circle, while for bubbles of unequal volumes (Fig. 1b), the surface is a spherical segment that bows into the larger bubble. In this paper, we consider only these standard [1] configurations. Nonstandard double bubbles, where one bubble envelops another, are always unstable. Recently, Hutchings et al. [1] proved the *double bubble conjecture* that the equilibrium standard double bubble “provides the least-area way to enclose and separate two regions of prescribed volume” in Euclidean 3-space, $\mathbf{R}^3$. The theorem states: *In* $\mathbf{R}^3$, *the unique perimeter-minimizing double bubble enclosing and separating regions* Ω_1 *and* Ω_2 *of prescribed volumes* v_1 *and* v_2 *is a standard double bubble consisting of three spherical caps meeting along a common circle at* 120° *angles*. (*For equal volumes, the middle cap is a flat disc*.) Details of the proof are presented elsewhere [2]. For the particular case of equal volumes, $v_1 = v_2$, the conjecture was proved by Hass et al. [3,4]. References to other work dealing with the double bubble problem can be found in Refs. [1,2].

It follows directly from the above theorem that an equilibrium double bubble is stable to perturbations that keep the volumes v_1 and v_2 fixed. Indeed, the double bubble’s potential energy, U_b, is proportional to the configuration’s total surface area. On the other hand, the energy U_b coincides with the system total energy, E, in the case of fixed v_1 and v_2. Consequently, the minimum of the film area under the constraint of fixed v_1 and v_2 is equivalent to the total energy minimum, and thus, to the stability of the double bubble.

Note, however, that the volumes v_1 and v_2 could be subject to perturbations that allow the volumes to vary. In this case, the expression for the total energy functional E accounts not only for U_b, but internal energy of the gas as well. The stability of double bubbles to general perturbations that may not be volume-conserving is examined in this paper. To study the stability, we assume that changes in the gas state occur such that the pressures and the volumes in the initial and final states are related through a polytropic process.

It may appear that the theorem of Hutchings et al. proves stability of a double *drop* formed by two immiscible incompressible liquids under zero gravity. In this case, the energy functional consists only of the surface potential energy $(E \equiv U_d)$. However, this potential energy is a linear function of areas of three interfaces (“liquid 1–gas,” “liquid 2–gas,” and “liquid 1–liquid 2”) with different factors that represent the corresponding interface tensions. The potential energy cannot be proportional to the total area of the interfaces.

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Thus, double drop stability does not necessarily follow from this theorem.

To investigate the stability of a double bubble to non-volume-conserving (nonisochoric) perturbations (see Section 2), we consider the sign of the minimum value of the second variation, $\delta^2 E$, of the bubble's total energy. This coincides with the sign of the smallest eigenvalue of the associated spectral problem for all admissible perturbations. The spectral problem is split up into problems related to axisymmetric perturbations and a sequence of problems related to nonaxisymmetric perturbations. According to the theorem outlined above, a double bubble is stable to nonaxisymmetric perturbations because they do not affect the volumes v_1 and v_2. Thus, the stability to axisymmetric perturbations remains to be investigated. The analysis of the corresponding spectral problem has been performed. The result is that the shape of the double bubble is stable to axisymmetric perturbations (translations of the bubble along its axis of symmetry are neglected). Consequently, the equilibrium double bubble is always stable.

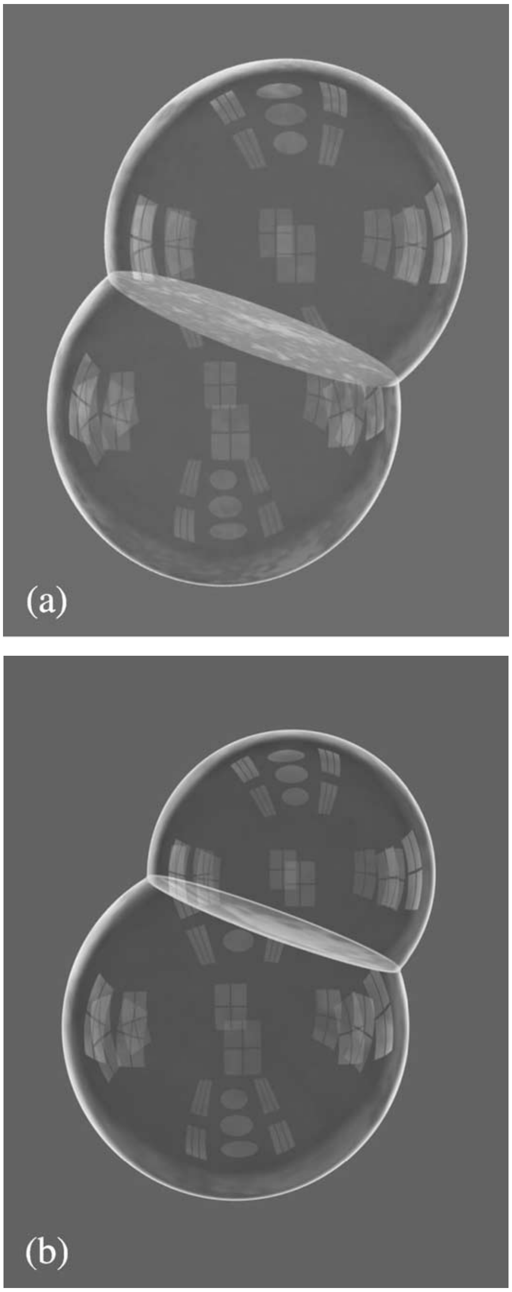

Fig. 1. (a) Equal volume and (b) unequal volume standard double bubble configurations. (Courtesy J.M. Sullivan, http://www.math.uiuc.edu/~jms/images/double/. Published with permission.)

As regards the stability of a double drop, we examine a variational formulation of the problem. The expression for $\delta^2 U_d$ and the associated spectral problem are presented in Section 3.

2. Equilibrium and stability of double bubbles

It is assumed that the bubble surfaces envelope the simply connected regions Ω_1 and Ω_2 that are occupied by gas and that the double bubble surfaces are surrounded by a third region Ω_3 occupied by the same gas (Fig. 2a). The liquid film separating these regions is assumed to be homogeneous. The interface between Ω_j and Ω_k (see Fig. 2a) is denoted by Γ_i $(i \neq j \neq k;\ i, j, k = 1, 2, 3)$, and the liquid–gas surface tension is σ.

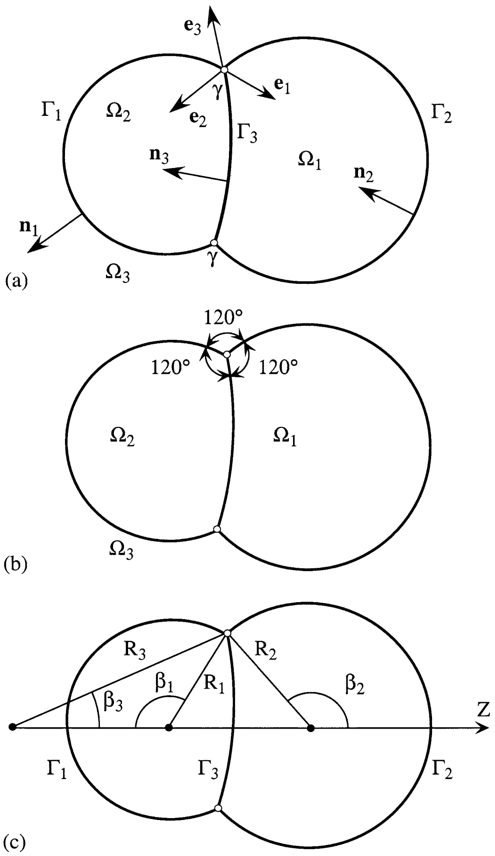

Fig. 2. Geometry of a double bubble.

2.1. Equilibrium conditions

The equilibrium conditions for such a double bubble configuration are obtained from the stationary total energy functional, E. The variation of E is

$$\delta E = \delta U_b - \sum_{i=1}^{3} P_i \delta v_i. \tag{1}$$

Here, U_b is the potential energy of a double bubble and P_i and v_i are, respectively, the gas pressures and volumes of the regions Ω_i $(i = 1, 2, 3)$ defined above. The quantities δv_i are the variations of the volumes v_i $(i = 1, 2, 3)$. If $|\Gamma_i|$ is the area of Γ_i, the potential energy U_b has the form

$$U_b = 2\sigma \sum_{i=1}^{3} |\Gamma_i|. \tag{2}$$

There are two "gas–liquid" interfaces for each film surface Γ_i, hence the factor 2 in Eq. (2) (see, for example, Refs. [5,6]). If each point on the surface Γ_i with a radius vector $\mathbf{r}$ is displaced by a small vector $\delta\mathbf{r}$, the first variation, δU_b, of the potential energy is [5]

$$\delta U_b = 2\sigma \sum_{i=1}^{3} \biggl(-\int\limits_{\Gamma_i} 2H_i \mathbf{n}_i \cdot \delta\mathbf{r}\, d\Gamma + \int\limits_{\gamma} \mathbf{e}_i \cdot \delta\mathbf{r}\, d\gamma \biggr). \tag{3}$$

Here, H_i is the mean curvature of the surface Γ_i; $\mathbf{n}_i$ is the unit vector normal to Γ_i; γ is the contact line of the surfaces Γ_1, Γ_2, and Γ_3; and $\mathbf{e}_i$ is a unit vector normal to γ that lies in the plane tangential to Γ_i and is directed outward from Γ_i (Fig. 2a). We assume that $\mathbf{n}_i$ is directed from Ω_j to Ω_k. For $i = 1$, we choose $j = 2$, $k = 3$. For other i, the values j and k are determined by cyclic permutation (see Fig. 2a). Direction of $\mathbf{n}_i$ is important for determination the sign of a curvature. Note that, the curvature of a normal section of Γ_i is taken to be positive if this section is concave in direction of $\mathbf{n}_i$, i.e. is convex toward the domain Ω_j.

The variation of each gas volume is

$$\delta v_i = -\int\limits_{\Gamma_j} \mathbf{n}_j \cdot \delta\mathbf{r}\, d\Gamma + \int\limits_{\Gamma_k} \mathbf{n}_k \cdot \delta\mathbf{r}\, d\Gamma$$
$$(i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2). \tag{4}$$

Finally, the equality $\delta E = 0$ leads to the following equilibrium conditions:

$$4\sigma H_i = -P_j + P_k \quad (\text{on } \Gamma_i)$$
$$(i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2), \tag{5}$$

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0 \quad (\text{on } \gamma). \tag{6}$$

2.2. Characteristics of the equilibrium configuration

Equation (6) indicates that surfaces Γ_i intersect at 120° angles (Fig. 2b). This property is known as Plateau's rule [5]. Equations (5) are the Young–Laplace equations. According to (5), the surfaces Γ_i are spherical segments with radii R_i $(i = 1, 2, 3)$ and it follows that

$$H_1 = -1/R_1, \qquad H_2 = 1/R_2,$$
$$H_3 = (R_2 - R_1)/R_1 R_2. \tag{7}$$

The last equality can also be represented as

$$H_3 = 1/R_3 \quad \text{if } R_2 \geqslant R_1 \quad \text{and}$$
$$H_3 = -1/R_3 \quad \text{if } R_2 \leqslant R_1. \tag{8}$$

Hereafter, we will assume that $R_2 \geqslant R_1$ (Fig. 2).

The volumes v_1 and v_2 determine all parameters of the system. We choose R_2 as a characteristic length and introduce the radius ratio

$$K = R_1/R_2. \tag{9}$$

Considering the relations

$$R_1 \sin\beta_1 = R_2 \sin\beta_2 = R_3 \sin\beta_3, \tag{10}$$

$$\beta_1 + \beta_2 = 4\pi/3, \tag{11}$$

$$\beta_2 - \beta_3 = 2\pi/3 \tag{12}$$

between radii R_1, R_2, R_3 and polar angles β_1, β_2, β_3 of the spherical segments Γ_1, Γ_2, Γ_3 (see Fig. 2c), we can express other parameters of the system as functions of K and R_2:

$$R_3 = R_2 K/(1-K), \tag{13}$$

$$\cos\beta_1 = f_1(K) = (1-2K)\big/\bigl(2\sqrt{1-K+K^2}\bigr), \tag{14}$$

$$\cos\beta_2 = f_2(K) = -(2-K)\big/\bigl(2\sqrt{1-K+K^2}\bigr), \tag{15}$$

$$\cos\beta_3 = f_3(K) = (1+K)\big/\bigl(2\sqrt{1-K+K^2}\bigr), \tag{16}$$

$$v_1 = \frac{1}{3}\pi R_2^3 \bigl\{(1-f_2)^2(2+f_2) - K^3(1-f_3)^2(2+f_3)/(1-K)^3\bigr\}, \tag{17}$$

$$v_2 = \frac{1}{3}\pi R_2^3 K^3 \bigl\{(1-f_1)^2(2+f_1) + (1-f_3)^2(2+f_3)/(1-K)^3\bigr\}. \tag{18}$$

For given v_1 and v_2, we can first calculate the values K and R_2 from (17) and (18), and then the radii R_1, R_3 and angles β_1, β_2, β_3 from (9), (13)–(16).

2.3. The second variation of the total energy

For a polytropic process we have $P_i v_i^n = \tilde{c}_i$, where $\tilde{c}_i$ is a constant and n is the polytropic exponent. Hence

$$\delta P_i = -\frac{nP_i}{v_i}\delta v_i. \tag{19}$$

Variation of the expression (1) for δE and accounting for the equality (19) and the equilibrium conditions (5), (6) yields

$$\delta^2 E = -2\sigma \sum_{i=1}^{3} \int\limits_{\Gamma_i} \delta(2H_i)\mathbf{n}_i \cdot \delta\mathbf{r}\, d\Gamma + 2\sigma \int\limits_{\gamma} \Biggl(\sum_{i=1}^{3} \delta\mathbf{e}_i\Biggr) \cdot \delta\mathbf{r}\, d\gamma + \sum_{i=1}^{3} \frac{nP_i}{v_i}(\delta v_i)^2. \tag{20}$$

We use the expression for $\delta(2H_i)$ presented in Ref. [7] and (A.7), the expression for $\delta\mathbf{e}_i$ developed in Appendix A, to get

$$\begin{aligned}\frac{1}{2\sigma}\delta^2 E = &-\sum_{i=1}^{3}\int_{\Gamma_i}\left(\Delta_i N_i + \frac{2}{R_i^2}N_i\right)N_i\,d\Gamma\\ &+\int_{\gamma}\left[\sum_{i=1}^{3}\left(\frac{\partial N_i}{\partial s_i} + H_i h_i\right)N_i\right]d\gamma\\ &+\frac{1}{2}\sum_{i=1}^{3}T_i(\delta v_i)^2. \qquad (21)\end{aligned}$$

Here,

$$N_i = \mathbf{n}_i\cdot\delta\mathbf{r}\quad(\mathbf{r}\in\Gamma_i),\qquad h_i = \mathbf{e}_i\cdot\delta\mathbf{r}\quad(\mathbf{r}\in\gamma),$$
$$T_i = nP_i/(\sigma v_i), \qquad (22)$$

Δ_i is the Laplace–Beltrami operator on Γ_i, and s_i is the arc length of the section of Γ_i orthogonal to γ measured in the direction $\mathbf{e}_i$. Finally, substituting the expression for h_i in terms of N_j and N_k

$$h_i = (-N_j + N_k)\cot(2\pi/3), \qquad (23)$$

and assuming that T_3 is negligible (because v_3 is infinite or large compared to v_1 and v_2), we obtain

$$\begin{aligned}\frac{1}{2\sigma}\delta^2 E = &-\sum_{i=1}^{3}\int_{\Gamma_i}\left(\Delta_i N_i + \frac{2}{R_i^2}N_i\right)N_i\,d\Gamma\\ &+\int_{\gamma}\left[\sum_{i=1}^{3}\left(\frac{\partial N_i}{\partial s_i} - \frac{\sqrt{3}}{3}H_i(-N_j + N_k)\right)N_i\right]d\gamma\\ &+\frac{1}{2}T_1(\delta v_1)^2 + \frac{1}{2}T_2(\delta v_2)^2. \qquad (24)\end{aligned}$$

2.4. *Spectral stability criterion*

According to the principle of minimum total energy, the equilibrium state of the system is stable if the minimum value of $\delta^2 E$ over all admissible perturbations is positive, and is unstable if it is negative [7]. We can show (see Appendix B) under the normalization condition

$$\sum_{i=1}^{3}\int_{\Gamma_i}N_i^2\,d\Gamma = 1 \qquad (25)$$

that the minimum value of the quadratic functional (24) is equal to the smallest eigenvalue $\lambda = \lambda^*$ of the following problem:

$$-\Delta_i N_i - \frac{2}{R_i^2}N_i + \xi_i = \lambda N_i \quad (\text{on } \Gamma_i,\ i = 1,2,3), \qquad (26)$$

$$\left(\frac{\partial N_1}{\partial s_1} + \chi_1 N_1\right) - \left(\frac{\partial N_3}{\partial s_3} + \chi_3 N_3\right) = 0 \quad (\text{on } \gamma), \qquad (27)$$

$$\left(\frac{\partial N_2}{\partial s_2} + \chi_2 N_2\right) - \left(\frac{\partial N_3}{\partial s_3} + \chi_3 N_3\right) = 0 \quad (\text{on } \gamma), \qquad (28)$$

$$N_1 + N_2 + N_3 = 0 \quad (\text{on } \gamma), \qquad (29)$$

$$\xi_1 = \frac{1}{2}T_2\delta v_2 = \frac{1}{2}T_2\left(-\int_{\Gamma_3}N_3\,d\Gamma + \int_{\Gamma_1}N_1\,d\Gamma\right), \qquad (30)$$

$$\xi_2 = -\frac{1}{2}T_1\delta v_1 = -\frac{1}{2}T_1\left(-\int_{\Gamma_2}N_2\,d\Gamma + \int_{\Gamma_3}N_3\,d\Gamma\right), \qquad (31)$$

$$\xi_3 = -\xi_1 - \xi_2. \qquad (32)$$

Here,

$$\chi_i = \frac{\sqrt{3}}{3}(H_j - H_k)$$
$$(i = 1,2,3;\ j = 2,3,1;\ k = 3,1,2). \qquad (33)$$

A double bubble configuration is stable if $\lambda^* > 0$, and is unstable if $\lambda^* < 0$. Note that, according to the boundary conditions (27), (28), the perturbed surfaces $\widetilde{\Gamma}_i$ intersect at 120° angles under perturbations corresponding to eigenfunctions of the spectral problem.

The spherical segments Γ_1, Γ_2, and Γ_3 are symmetric about the z-axis (Fig. 2c). We choose the azimuthal angle θ_i and the arc length s_i of the axial cross section as curvilinear coordinates on spherical segment Γ_i. The operator Δ_i on Γ_i then assumes the form

$$\Delta_i = \frac{\partial^2}{\partial s_i^2} + \frac{1}{R_i}\cot\left(\frac{s_i}{R_i}\right)\frac{\partial}{\partial s_i} + \frac{1}{R_i^2}\frac{\partial^2}{\partial\theta_i^2}. \qquad (34)$$

The search for solutions in the form

$$N_i = \psi_{i0}(s_i) + \sum_{m=1}^{\infty}\left[\psi_{im}(s_i)\cos(m\theta_i) + \varphi_{im}(s_i)\sin(m\theta_i)\right] \qquad (35)$$

leads to a sequence of one-dimensional boundary-value problems for ψ_{i0}, ψ_{im}, and φ_{im} $(m \geqslant 1)$. The following problem corresponds to axisymmetric perturbations:

$$-\psi_{i0}'' - \frac{1}{R_i}\cot\left(\frac{s_i}{R_i}\right)\psi_{i0}' - \frac{2}{R_i^2}\psi_{i0} + \xi_i = \lambda\psi_{i0}$$
$$\left(' = \frac{d}{ds_i};\ 0 \leqslant s_i \leqslant R_i\beta_i;\ i = 1,2,3\right), \qquad (36)$$

$$(\psi_{10}' + \chi_1\psi_{10})_{s_1 = R_1\beta_1} - (\psi_{30}' + \chi_3\psi_{30})_{s_3 = R_3\beta_3} = 0, \qquad (37)$$

$$(\psi_{20}' + \chi_2\psi_{20})_{s_2 = R_2\beta_2} - (\psi_{30}' + \chi_3\psi_{30})_{s_3 = R_3\beta_3} = 0, \qquad (38)$$

$$\psi_{10}(R_1\beta_1) + \psi_{20}(R_2\beta_2) + \psi_{30}(R_3\beta_3) = 0, \qquad (39)$$

$$\begin{aligned}\xi_1 = \pi T_2\Bigg(&-R_3\int_0^{R_3\beta_3}\psi_{30}(s_3)\sin\frac{s_3}{R_3}\,ds_3\\ &+R_1\int_0^{R_1\beta_1}\psi_{10}(s_1)\sin\frac{s_1}{R_1}\,ds_1\Bigg), \qquad (40)\end{aligned}$$

$$\xi_2 = -\pi T_1\left(-R_2\int\limits_0^{R_2\beta_2}\psi_{20}(s_2)\sin\frac{s_2}{R_2}\,ds_2 + R_3\int\limits_0^{R_3\beta_3}\psi_{30}(s_3)\sin\frac{s_3}{R_3}\,ds_3\right), \tag{41}$$

$$\xi_3 = -\xi_1 - \xi_2. \tag{42}$$

The sign of the smallest eigenvalue λ_{01} of this problem determines whether the system is stable ($\lambda_{01} > 0$) or unstable ($\lambda_{01} < 0$) to axisymmetric perturbations. The problems for the functions $\psi_{im}(s_i)$ and $\varphi_{im}(s_i)$ ($i = 1, 2, 3$; $m = 1, 2, \ldots$) that correspond to nonaxisymmetric perturbations need not be considered. According to the theorem proved in Refs. [1] and [2], a double bubble is stable to such perturbations because they are volume-preserving, i.e., $\delta v_1 = \delta v_2 = 0$. The latter result follows from Eq. (4) written in the form

$$\delta v_i = -\int\limits_0^{2\pi}\int\limits_0^{R_j\beta_j} N_j(\theta_j, s_j)r_j(s_j)\,d\theta_j\,ds_j + \int\limits_0^{2\pi}\int\limits_0^{R_k\beta_k} N_k(\theta_k, s_k)r_k(s_k)\,d\theta_k\,ds_k, \tag{43}$$

where $i = 1, 2, 3$; $j = 2, 3, 1$; $k = 3, 1, 2$; and N_i is given by Eq. (35).

2.5. Analysis of the spectral problem

To investigate an existence of critical state ($\lambda_{01} = 0$) we take $\lambda = 0$ in Eqs. (36). The solutions of (36) for $\lambda = 0$ are

$$N_i = C_i\cos\left(\frac{s_i}{R_i}\right) + \xi_i\frac{R_i^2}{2} \quad (i = 1, 2, 3;\ C_i = \text{const}). \tag{44}$$

Substituting the expressions (44) into (37)–(41) and using (42) together with

$$T_2 = T_1(v_1/v_2)^{n+1} \tag{45}$$

yields the following system of equations for C_1, C_2, C_3, ξ_1, and ξ_2:

$$-C_1 + C_3 - \left(\xi_1R_1^2\right)\frac{1}{2}\left[f_1 + \frac{1}{(1-K)^2}f_3\right] - \left(\xi_2R_1^2\right)\frac{1}{2(1-K)^2}f_3 = 0, \tag{46}$$

$$-C_2 + C_3 - \left(\xi_1R_1^2\right)\frac{1}{2(1-K)^2}f_3 - \left(\xi_2R_1^2\right)\frac{1}{2}\left[\frac{1}{K^2}f_2 + \frac{1}{(1-K)^2}f_3\right] = 0, \tag{47}$$

$$C_1f_1 + C_2f_2 + C_3f_3 + \left(\xi_1R_1^2\right)\frac{1}{2}\left[1 - \frac{1}{(1-K)^2}\right] + \left(\xi_2R_1^2\right)\frac{1}{2}\left[\frac{1}{K^2} - \frac{1}{(1-K)^2}\right] = 0, \tag{48}$$

$$C_1 - C_3 + \left(\xi_1R_1^2\right)\frac{4}{3}\left[1 - f_1 + \frac{1}{(1-K)^4}(1-f_3) - \frac{2}{\pi TK^4f_5^{n+1}}\right] + \left(\xi_2R_1^2\right)\frac{4}{3}\frac{1}{(1-K)^4}f_4(1-f_3) = 0, \tag{49}$$

$$C_2 - C_3 + \left(\xi_1R_1^2\right)\frac{4}{3}\frac{1}{(1-K)^4}f_4(1-f_3) + \left(\xi_2R_1^2\right)\frac{4}{3}f_4\left[\frac{1}{K^4}(1-f_2) + \frac{1}{(1-K)^4}(1-f_3) - \frac{2}{\pi TK^4}\right] = 0. \tag{50}$$

Here, the following notation has been used:

$$f_4(K) = 1 - K + K^2, \qquad f_5(K) = v_1/v_2, \tag{51}$$

$$T = R_2^4T_1. \tag{52}$$

The condition that a nontrivial solution of (46)–(50) exists is identically satisfied because Eqs. (46)–(48) are linearly dependent. That is, $\lambda = 0$ is always an eigenvalue for problem (36)–(42) independent of K, T, and n. This eigenvalue corresponds to axisymmetric perturbations that cause a rigid body translation of a double bubble along the z-axis. Clearly, the double bubble equilibrium is indifferent to such a translation. In order to ignore displacements along the z-axis and consider the stability of the *shape* of a double bubble we impose the following restriction: Perturbations must leave the z-coordinate of the double bubble's mass center unchanged. That is,

$$\delta\int\limits_{\Omega_1\cup\Omega_2} z\,d\Omega = \int\limits_{\Gamma_1} zN_1\,d\Gamma - \int\limits_{\Gamma_2} zN_2\,d\Gamma = 0. \tag{53}$$

Together with Eq. (44) the above condition takes the form

$$C_1K^3\left[\frac{1}{2}\left(1-f_1^2\right) - \frac{1}{3}\left(1-f_1^3\right)\right] + C_2\left[\frac{1}{2}\left(1-f_2^2\right) - \frac{1}{3}\left(1-f_2^3\right)\right] + \left(\xi_1R_1^2\right)K^3\left[\frac{1}{2}(1-f_1) - \frac{1}{4}\left(1-f_1^2\right)\right] + \left(\xi_2R_1^2\right)\frac{1}{K^2}\left[\frac{1}{2}(1-f_2) - \frac{1}{4}\left(1-f_2^2\right)\right] = 0. \tag{54}$$

To obtain a system of five linearly independent equations with respect to C_1, C_2, C_3, ξ_1, and ξ_2, we replace one of the equations (46)–(48), say (48), with the condition (54). Let $D(K, T, n)$ be the determinant of the resulting system. Then the equality

$$D(K, T, n) = 0 \tag{55}$$

is the necessary and sufficient condition for some eigenvalue of problem (36)–(42) with the constraint (53) to be equal to zero. For given K and n, (55) is a quadratic equation that gives two possible values of T that correspond to $\lambda = 0$.

Table 1
Critical values of the parameter T for $n = 1$ and different radius ratios K

K	T^*
0^+	0.3183
0.1	0.3183
0.2	0.3183
0.3	0.3184
0.4	0.3188
0.5	0.3197
0.6	0.3216
0.7	0.3250
0.8	0.3309
0.9	0.3422
1.0	0.3537

Since λ_{01} increases with T (see expression (24) for $\delta^2 E$), the larger root ($T = T^*$) corresponds to $\lambda_{01} = 0$, and we conclude that the double bubble is stable to axisymmetric perturbations if $T > T^*$, and is unstable if $T < T^*$.

A polytropic process is generally nonisothermal. For an ideal gas, $P_i v_i = \widetilde{N}_i R_u \Theta_i$ (here, $\widetilde{N}_i$ and Θ_i are the number of moles and the temperature in Ω_i, and R_u is the universal gas constant), the process must be isothermal at $n = 1$, while for $n \neq 1$ the process is nonisothermal. For the system under consideration, the region Ω_3 with the relatively large volume v_3 serves as a thermal reservoir. Thus, any deviation from the equilibrium temperatures Θ_1 or Θ_2 will lead to heat transfer between regions Ω_1 or Ω_2 and Ω_3, respectively. The question is—how rapidly does the process associated with the perturbation occur? A process that occurs slowly will be isothermal. For an extremely rapid process, where there is no time to transfer heat from the interior of either bubble, the process is essentially adiabatic. Thus, for the system $\Omega = \bigcup_{i=1}^{3} \Omega_i$ under consideration, the exponent n lies in the interval $1 \leqslant n \leqslant \eta$, where $\eta = c_p/c_v$ is the adiabatic exponent, and c_p and c_v are the gas-specific heats at constant pressure and volume, respectively.

It follows from (24) and (22) that λ_{01} increases with n. We consider the case $n = 1$ that corresponds to the lowest level of stability. The cumbersome expression for D and roots T^* of (55) at fixed values of K, $0 < K \leqslant 0.9$, and $n = 1$ were obtained using Mathematica [8]. Values of T^* are presented in Table 1. Two of these values (for $K = 0.5$ and 0.9) were verified by independently calculating all the elements of the determinant for a given K, then computing the coefficients in the expression for $D(T)$, and finally calculating the roots of the equation $D(T) = 0$.

Equation (55) has a singularity at $K = 1$. Therefore, the case of equidimensional bubbles, $K = 1$, was analyzed separately. In this case, all five algebraic equations assume the form $\xi_1 + \xi_2 = 0$. To find value of T^*, we substitute $N_1 = \xi_1 R_2^2/2$, $N_2 = -\xi_1 R_2^2/2$, and $N_3 = 0$ into (24) and use the fact that $T_1 = T_2$. Then the equality $\delta^2 E = 0$ reduces to

$$T^* = 10/(9\pi) = 0.35368. \tag{56}$$

As the data in Table 1 suggest, this value, obtained analytically, is the limit for the sequence of calculated T^* as $K \to 1$. (This might be considered as circumstantial evidence that our approach is valid.) Note that the critical value of T for $K = 1$ determined by (56) is valid for any n.

It now remains for us to compute T and compare it with T^*. We get

$$T = R_2^4 \frac{n P_1}{\sigma v_1} = \frac{n}{\varpi}\left(4 + \frac{P_3 R_2}{\sigma}\right). \tag{57}$$

The value of $\varpi = v_1/R_2^3$ decreases from $4\pi/3$ to $9\pi/8$ when K increases from 0^+ to 1. Since $P_3 > 0$, we have $T > 0.9549$ for $K > 0$, and we can conclude that $T > T^*$ for $0 < K \leqslant 1$ for $n = 1$. This means that the double bubble is stable to axisymmetric perturbations for any K and any n ($1 \leqslant n \leqslant \eta$). From this result and the result published in Ref. [1] we can conclude that the double bubble is always stable.

3. Double drops

For weightless double drops we will use the notation introduced in Fig. 2a for double bubbles except that domains Ω_1 and Ω_2 are now occupied by immiscible incompressible liquids and are surrounded by a gas (domain Ω_3). The equilibrium and stability conditions follow from stationary and minimum principles of the total energy that coincides with the potential energy of drops,

$$U_d = \sum_{i=1}^{3} \sigma_i |\Gamma_i|. \tag{58}$$

Here, σ_i is the surface tension on Γ_i.

Since the liquids are incompressible, volumes v_1, v_2, and v_3 of the regions Ω_1, Ω_2, and Ω_3 remain unchanged under perturbation, so that

$$\delta v_i \equiv -\int_{\Gamma_j} N_j \, d\Gamma + \int_{\Gamma_k} N_k \, d\Gamma = 0$$
$$(i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2). \tag{59}$$

Equations (59) impose restrictions on the admissible perturbations $\delta \mathbf{r}$. From the variation of the potential energy and the constant volume constraint for v_i we get

$$\delta U_d - \sum_{i=1}^{3} P_i \delta v_i = 0, \tag{60}$$

where P_i is the Lagrange multiplier that represents the pressure in the domain Ω_i. Equation (60) leads to the following equilibrium conditions:

$$\sigma_i 2H_i = -P_j + P_k \quad (\text{on } \Gamma_i)$$
$$(i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2) \tag{61}$$

$$\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 + \sigma_3 \mathbf{e}_3 = 0 \quad (\text{on } \gamma). \tag{62}$$

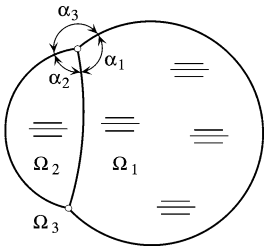

Fig. 3. Double drop geometry.

Equation (62) can be satisfied only if

$$\sigma_i < \sigma_j + \sigma_k \quad (i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2). \tag{63}$$

When the conditions (63) are fulfilled, Eq. (62) determines the dihedral angles α_i formed by domains Ω_i along the contact line γ (Fig. 3),

$$\cos\alpha_i = \frac{\sigma_i^2 - \sigma_j^2 - \sigma_k^2}{2\sigma_j\sigma_k} \quad (i = 1, 2, 3;\ j = 2, 3, 1;\ k = 3, 1, 2). \tag{64}$$

Conditions (64) are valid for any equilibrium configuration involving the common contact line between three immiscible fluids (one of them may be a gas); see Ref. [7]. Dihedral angles α_i depend on values of surface tension on all three interfaces Γ_i, unlike the case of double bubbles where $\alpha_i = 2\pi/3$.

According to (62), surfaces Γ_i are spherical segments of some radii R_i $(i = 1, 2, 3)$, and

$$H_1 = -1/R_1, \qquad H_2 = 1/R_2,$$
$$H_3 = (\sigma_1 R_2 - \sigma_2 R_1)/\sigma_3 R_1 R_2. \tag{65}$$

The last equality can also be rewritten in the form

$$H_3 = \frac{1}{R_3} \quad \text{if } (\sigma_1 R_2 - \sigma_2 R_1) > 0 \quad \text{and}$$
$$H_3 = -\frac{1}{R_3} \quad \text{if } (\sigma_1 R_2 - \sigma_2 R_1) < 0. \tag{66}$$

Instead of relations (11), (12), (17), (18), we get respectively

$$\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = 0, \tag{67}$$

$$\beta_2 - \beta_3 - \alpha_1 = 0 \quad (\text{if } H_3 > 0),$$
$$\beta_2 + \beta_3 - \alpha_1 = 0 \quad (\text{if } H_3 < 0), \tag{68}$$

$$v_1 = \frac{1}{3}\pi R_2^3 \biggl\{ (1 - \cos\beta_2)^2 (2 + \cos\beta_2) - \frac{\sin^3\beta_2}{\sin^3(\beta_2 - \alpha_1)} \bigl[1 - \cos(\beta_2 - \alpha_1)\bigr]^2 \times \bigl[2 + \cos(\beta_2 - \alpha_1)\bigr] \biggr\}, \tag{69}$$

$$v_2 = \frac{1}{3}\pi R_2^3 \sin^3\beta_2 \biggl\{ \frac{1}{\sin^3(\alpha_1 + \alpha_2 - \beta_2)} \times \bigl[1 - \cos(\alpha_1 + \alpha_2 - \beta_2)\bigr]^2 \times \bigl[2 + \cos(\alpha_1 + \alpha_2 - \beta_2)\bigr] + \frac{1}{\sin^3(\beta_2 - \alpha_1)} \bigl[1 - \cos(\beta_2 - \alpha_1)\bigr]^2 \times \bigl[2 + \cos(\beta_2 - \alpha_1)\bigr] \biggr\}. \tag{70}$$

For given α_1, α_2, v_1, and v_2, we can calculate the values R_2 and β_2 from (69), (70), and then the values β_1, R_1, β_3, and R_3 from (10), (67), (68).

The second variation of the potential energy is

$$\delta^2 U = -\sum_{i=1}^{3} \sigma_i \int_{\Gamma_i} \biggl(\Delta_i N_i + \frac{2}{R_i^2} \biggr) N_i \, d\Gamma + \int_{\gamma} \biggl\{ \sum_{i=1}^{3} \sigma_i \biggl[\frac{\partial N_i}{\partial s_i} + \frac{H_i}{\sin\alpha_i} \times (-N_j \cos\alpha_j + N_k \cos\alpha_k) \biggr] N_i \biggr\} d\gamma. \tag{71}$$

The spectral problem associated with the minimum of the functional (71) under the conditions (25) and (59) assumes the form

$$-\Delta_i N_i - \frac{2}{R_i^2} N_i + \mu_i = \lambda N_i \quad (\text{on } \Gamma_i,\ i = 1, 2, 3), \tag{72}$$

$$\biggl(\frac{\partial N_1}{\partial s_1} + b_1 N_1 \biggr) - \biggl(\frac{\partial N_3}{\partial s_3} + b_3 N_3 \biggr) = 0 \quad (\text{on } \gamma), \tag{73}$$

$$\biggl(\frac{\partial N_2}{\partial s_2} + b_2 N_2 \biggr) - \biggl(\frac{\partial N_3}{\partial s_3} + b_3 N_3 \biggr) = 0 \quad (\text{on } \gamma), \tag{74}$$

$$\sigma_1 N_1 + \sigma_2 N_2 + \sigma_3 N_3 = 0 \quad (\text{on } \gamma) \tag{75}$$

with the additional requirements (59). Here we have used the notation

$$b_i = \frac{1}{2} \biggl(\frac{H_i}{\sin\alpha_i} \frac{\sigma_j^2 - \sigma_k^2}{\sigma_j \sigma_k} + \frac{H_j}{\sin\alpha_j} \frac{\sigma_j}{\sigma_k} - \frac{H_k}{\sin\alpha_k} \frac{\sigma_k}{\sigma_j} \biggr). \tag{76}$$

In contrast to problem (26)–(32) where ξ_i $(i = 1, 2, 3)$ are expressed by relations (30)–(32), unknown constants μ_i entering into (72) are determined from the conditions (59). Here, $\mu_1 + \mu_2 + \mu_3 = 0$ and of the three conditions (59) it is sufficient to consider two.

With the expansion (35), the stability problem can be reduced to a set of spectral problems. It suffices to analyze three of them. These problems are associated with the following types of perturbations: axisymmetric perturbations that satisfy the condition (53), first-harmonic perturbations $(m = 1)$ that leave the mass center on the z-axis, and second-harmonic perturbations $(m = 2)$.

4. Concluding remarks

We have shown that the shape of an equilibrium double bubble is always stable under perturbations that can change the volume of each of the two bubbles that are the components of the double bubble. If we also consider the recently proven theorem that a standard double bubble is stable to perturbations that preserve the volume of each component (Refs. [1,2]), then our result leads to the conclusion that the shape of a standard double bubble is linearly stable to arbitrary perturbations. The observed relatively short lifetime of the double bubble can be attributed to the same reasons that a single soap bubble or film is short-lived. Many factors (drainage, evaporation, diffusion of the gas through film surfaces, the surfactant concentration, etc.) may cause the double bubble to collapse (see, for example, Ref. [5] for details).

Formally, a double bubble is stable for any polytropic process with $n \geqslant 1$. However, the conclusion on stability cannot be extended to the more general case of $n \geqslant 0$. Our calculations for $n = 0$ have shown that $T^*(K) > 0$ $(0 < K \leqslant 1)$. Since $T(K) = 0$ at $n = 0$ (see (57)), a double bubble is unstable for $n = 0$. This means that a double bubble will be unstable for any polytropic process with $n \leqslant 0$. For polytropic processes with $0 < n < 1$, equidimensional bubbles $(K = 1)$ are stable provided $n > 5/(16 + 4P_3R_2/\sigma)$ (see (56), (57)). Note, however, that for the system under consideration, polytropic processes with n outside the interval $1 \leqslant n \leqslant \eta$ are physically unfeasible.

The stability analysis was based on the principle of minimum total energy E and was realized through an analysis of the sign of the minimum value of the second variation $\delta^2 E$ over all admissible perturbations. The associated spectral problem employed for this purpose was solved numerically. It follows from the published result [1,2] that the most dangerous perturbation is one that takes an original double bubble to another equilibrium double bubble corresponding to the new values of the volumes v_1 and v_2. However, the direct calculation of $\delta^2 E$ over such a perturbation leads to an expression too cumbersome for analytical analysis and so would require numerical analysis. Because of this, we adopted the general method described in the paper.

The same method was used to investigate the closely related stability problem for a double drop under zero gravity. In this case, the type of dangerous perturbation cannot be predetermined. The mathematical formulation of the problem for general perturbations was presented, and the possible method of solution was outlined. The solution of this problem in the general case remains open because a double drop system has many parameters (much larger than those for a double bubble system). However, for *specific* values of the parameters, the solution can be obtained.

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Appendix A. Variation of the vector $\mathbf{e}_i$

We denote the unit vector of the tangent to γ by

$$\mathbf{t} = \mathbf{n}_i \times \mathbf{e}_i, \tag{A.1}$$

and consequently

$$\delta\mathbf{e}_i = (\delta\mathbf{t} \times \mathbf{n}_i) + (\mathbf{t} \times \delta\mathbf{n}_i). \tag{A.2}$$

There is a correspondence between the equilibrium and perturbed interfaces such that at each point $\mathbf{r} \in \gamma$ the vector $\delta\mathbf{r}$ will belong to the plane orthogonal to γ, i.e. $\mathbf{t} \cdot \delta\mathbf{r} = 0$. In the variation of a vector $\mathbf{A}_i$ that is defined on γ and relates to Γ_i, we take

$$\delta\mathbf{r} = \delta_1\mathbf{r} + \delta_2\mathbf{r}, \qquad \delta_1\mathbf{r} = N_i\mathbf{n}_i, \qquad \delta_2\mathbf{r} = h_i\mathbf{e}_i \tag{A.3}$$

and accordingly

$$\delta\mathbf{A}_i = \delta_1\mathbf{A}_i + \delta_2\mathbf{A}_i. \tag{A.4}$$

We have

$$\delta_1\mathbf{t} = \frac{\partial N_i}{\partial\tau}\mathbf{n}_i, \qquad \delta_1\mathbf{n}_i = -\frac{\partial N_i}{\partial s_i}\mathbf{e}_i - \frac{\partial N_i}{\partial\tau}\mathbf{t},$$
$$\delta_1\mathbf{e}_i = \frac{\partial N_i}{\partial s_i}\mathbf{n}_i, \tag{A.5}$$

$$\delta_2\mathbf{t} = \frac{\partial h_i}{\partial\tau}\mathbf{e}_i + h_i\left(\frac{\partial\mathbf{e}_i}{\partial\tau}\mathbf{n}_i\right)\mathbf{n}_i, \qquad \delta_2\mathbf{n}_i = -H_ih_i\mathbf{e}_i,$$
$$\delta_2\mathbf{e}_i = -\frac{\partial h_i}{\partial\tau}\mathbf{t} + H_ih_i\mathbf{n}_i, \tag{A.6}$$
$$\delta\mathbf{e}_i = -\frac{\partial h_i}{\partial\tau}\mathbf{t} + \left(H_ih_i + \frac{\partial N_i}{\partial s_i}\right)\mathbf{n}_i. \tag{A.7}$$

Here, τ and s_i are the arc lengths of the line γ and of the section of Γ_i normal to γ measured in directions of $\mathbf{t}$ and $\mathbf{e}_i$, respectively. It is taken into account in the second formula in (A.6) (and consequently in the third formula in (A.6) and in (A.7)) that Γ_i represents a spherical segment.

Appendix B. The minimum value of $(2\sigma)^{-1}\delta^2 E$

The minimization of the functional (24) under restriction (25) leads to the equation

$$\sum_{i=1}^{3}\int_{\Gamma_i}\left[\left(-\Delta_iN_i - \frac{2}{R_i^2}N_i\right) - \lambda N_i\right]\delta N_i\,d\Gamma$$

$$+\frac{1}{2}\int\limits_{\gamma}\left\{\sum_{i=1}^{3}\left[2\frac{\partial N_i}{\partial s_i}-\frac{\sqrt{3}}{3}H_i(-N_j+N_k)-\frac{\sqrt{3}}{3}(H_jN_j-H_kN_k)\right]\delta N_i\right\}d\gamma+\frac{1}{2}T_2\delta v_2\int\limits_{\Gamma_1}\delta N_1\,d\Gamma-\frac{1}{2}T_1\delta v_1\int\limits_{\Gamma_2}\delta N_2\,d\Gamma+\frac{1}{2}(T_1\delta v_1-T_2\delta v_2)\int\limits_{\Gamma_3}\delta N_3\,d\Gamma=0, \quad \text{(B.1)}$$

where λ is the Lagrange multiplier corresponding to condition (25). It follows from the equation $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3=0$, equivalent to (6), that on γ the quantities N_i, and also δN_i, are connected by the relations (29) and

$$\delta N_1+\delta N_2+\delta N_3=0. \quad \text{(B.2)}$$

Expressing δN_3 in terms of δN_1 and δN_2, and using (29), we represent the integral with respect to γ that occurs in (B.1) in the form

$$2\int\limits_{\gamma}\left[\left(\frac{\partial N_1}{\partial s_1}+\chi_1N_1\right)-\left(\frac{\partial N_3}{\partial s_3}+\chi_3N_3\right)\right]\delta N_1\,d\gamma+2\int\limits_{\gamma}\left[\left(\frac{\partial N_2}{\partial s_2}+\chi_2N_2\right)-\left(\frac{\partial N_3}{\partial s_3}+\chi_3N_3\right)\right]\delta N_2\,d\gamma, \quad \text{(B.3)}$$

with χ_i values given by Eq. (33). The spectral problem (26)–(32) then follows from (B.1)–(B.3).

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