

Stability of disconnected free surfaces in a cylindrical container under zero gravity: Simple cases

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The stability of equilibrium configurations of a capillary liquid in a circular cylindrical container with planar ends is investigated. The liquid is under zero gravity conditions, and its wetting angle is constant over the entire solid surface. Attention is focused on the case for which the free surface consists of two disconnected pieces (connectivity components) that bound the connected liquid domain. First we outline the method used to determine critical states with disconnected free surfaces when each connectivity component is axisymmetric. Then we examine the stability of disconnected surfaces for the simple cases that arise when each connectivity component represents a closed sphere or a part of a sphere. Ten configurations were considered that represent all possible combinations of the following connectivity components: A closed sphere (that bounds a gas bubble), a spherical cap in contact with the lateral wall of a cylinder; a spherical cap in contact with a cylinder endwall, and a portion of a sphere (that does not cross the cylinder's axis of symmetry) bounded by a cylindrical wall and a flat endwall. © 2000 American Institute of Physics. [S1070-6631(00)51311-7]

I. INTRODUCTION

Despite the fact that cylindrical containers are simple and widely used, knowledge of the shape and stability of an equilibrium free surface of a capillary liquid confined within a cylinder is limited. The first result directly linked to this problem was obtained by Concus.¹ It is related to the problem of the stability of a liquid suspended in a rectangular vertical channel and subject to a vertical gravity field. The analogous problem for an axisymmetric free surface of a liquid suspended in a circular cylindrical container was solved independently by Tyuptsov² (see also Ref. 3, pp. 147–150) and by Reynolds and Satterlee.⁴ They considered equilibrium states with *simply connected* axisymmetric free surfaces (i.e., surfaces that cross the container's symmetry axis) and obtained the value of the critical Bond number (the ratio of gravity forces to surface tension forces) as a function of the wetting angle. In particular, it was found that such a surface is *always* stable under zero gravity conditions. Later Concus⁵ carried out a sophisticated analysis of the shapes of stable simply connected axisymmetric surfaces. Tegart⁶ constructed examples of three-dimensional stable equilibrium shapes in a flat-bottomed cylindrical container for zero contact angle and for cases where gravity acts parallel to and perpendicular to the cylinder axis.

The stability of *doubly connected* axisymmetric free surfaces in a cylinder (a doubly connected surface consists of one piece and does not cross the symmetry axis) was recently considered by Slobozhanin *et al.*^{7,8} A free surface that has one contact line on one of the flat endwalls of a cylinder and

the other on the cylinder's lateral wall may exist in an axial gravity field. Under zero gravity the configuration may be stable or unstable. However, a particular configuration with a free surface that conforms to a piece of sphere and is bounded by a cylindrical wall and a flat endwall is always stable. A free surface with both contact lines on a cylindrical wall is possible only under zero gravity. Clearly, such a surface cannot have a spherical shape.

Thus, the stability problem has been studied for a connected axisymmetric free surface. Such a surface is typically an interface between a connected liquid region and connected gaseous region. However, a free surface can be *disconnected*, i.e., it may consist of several pieces (the connectivity components). There are two quite different types of disconnectivity. The first is when a configuration consists of several disconnected liquid regions, but each region is bounded by a separate free surface. In this case, the equilibrium and stability problem reduces to the problem for each separate free surface, i.e., to a set of problems for each connected free surface.

On the other hand, a free surface may be disconnected with $m > 1$ connectivity components Γ_i ($i = 1, \dots, m$) while the liquid region is connected [see, for example, Fig. 1(a) where $m = 2$]. This is the second type of disconnectivity. The stability problem for such configurations is more complicated than for a connected free surface (see Sec. II for an explanation of the fundamental differences) and to date has only been solved for a few model examples (Ref. 3, pp. 216–218).

Connected liquid regions bounded by disconnected free surfaces arise in a variety of situations, especially under microgravity conditions. For example, equilibrium liquids with gaseous inclusions develop after damping of a disturbance

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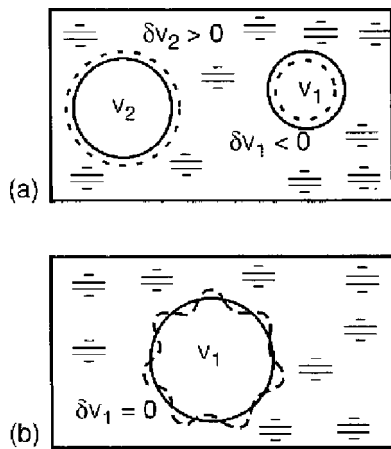


FIG. 1. (a) Configuration I (two connectivity components) in comparison with (b) a system with a single connectivity component. Dashed lines correspond to perturbed free surfaces.

that results in splashing of liquid propellant in a fuel tank. Such disturbances may arise as a consequence of transient vibration, abrupt changes in spacecraft orientation or even a sudden transition to weightlessness following a sustained thrust. Alternatively, voids and loss of a full contact between a melt and a cylindrical ampoule can occur during the isothermal soaking stage of low gravity directional crystallization experiments. The general mechanisms that might lead to such conditions have been discussed in Refs. 9 and 10 (see also references therein).

In this paper we analyze the stability of weightless liquid bounded by disconnected free surface consisting of two connectivity components ($m=2$) in a cylindrical container with flat bottoms. All possible configurations with connectivity components that represent a closed sphere or conform to a portion of a sphere are considered. It is assumed that the wetting angle α is constant over the entire solid surface (cylinder wall and two planar ends), and the physical properties of the gas are uniform for different gas cavities.

II. FORMULATION OF THE STABILITY PROBLEM

When each connectivity component of a free surface is axisymmetric, the stability of the system with respect to non-axisymmetric perturbations reduces to the determination of the stability of each connectivity component with respect to these perturbations. This is because nonaxisymmetric perturbations of separate connectivity components have no effect on the volumes of regions separated by these components. However, because liquid volume must be conserved, the problem of the system stability with respect to axisymmetric perturbations cannot be reduced to separate stability problems for each connectivity component. The only admissible axisymmetric perturbations are those which leave the volume of the liquid unchanged. For example, if a connected liquid region is bounded by a connected free surface, then only zero-volume change perturbations to this region are admissible. Thus, the variation δv_i of the gas cavity volume v_i

must be equal to zero. On the other hand, if a liquid region is bounded by a disconnected free surface, the total liquid volume must still be conserved

$$\sum_{i=1}^m \delta v_i = 0, \quad (1)$$

but each connectivity component need not satisfy a separate volume conservation condition. In other words, variations δv_i ($i=1,2,\dots,m$) of individual gas cavity volumes may differ from zero. For example, the volume of each bubble shown in Fig. 1(a) may be changed when perturbed (see dashed lines), but the volume of a single bubble [Fig. 1(b)] must be kept constant. Consequently, the class of admissible axisymmetric perturbations on each surface component Γ_i ($i=1,2,\dots,m$) is broader. Hence, the “level of stability” of the system with a disconnected free surface may be lower in comparison to a system that has only one of the connectivity components as its sole free surface. For example, the stability of the system ($i=2$) with connectivity components Γ_1 and Γ_2 may be lower than either of the systems that has Γ_1 , or Γ_2 , as its only free surface.

However, it should be recognized that perturbations of a free surface connectivity component that change the volume of an individual gas cavity, also change the internal energy of the gas and liquid phases. In contrast to the destabilizing effect due to the increase in breadth of admissible perturbations noted above, consideration of these internal energy changes leads to a stabilization effect. It is readily apparent however, that this stabilizing effect cannot lead to a greater level of stability than a system subject to the constraint that $\delta v_i = 0$, for each $i=1,2,\dots,m$.

Our analysis of the equilibrium stability for a system with a disconnected free surface is based on the basic principle of minimum potential energy postulated by Lagrange and Lyapunov (see, for example, Ref. 3, pp. 120–125). Accordingly, a stable equilibrium corresponds to a local minimum of potential energy.

For the equilibrium of non-rotating systems with capillary liquids it has been proven rigorously that “static” stability of a configuration corresponding to a minimum of potential energy is equivalent to the stability of that configuration obtained through dynamic analysis of the system’s stability. The latter involves perturbing the equations of motion. For the system under investigation here, it therefore suffices to examine perturbations of the equilibrium shapes rather than perturb the equations of motion (of which the equilibrium equations are a subset). Furthermore, for isothermal capillary liquids, an analysis of the system’s equilibrium can be accomplished far more easily through analysis of energy variations under admissible perturbations of free surfaces within the system, than through an analysis of solutions of the perturbed equations of motion. The physical validity of such an approach to the equilibrium stability of capillary liquids is well-established experimentally and has been the subject of numerous publications, including the early work of Plateau¹¹ and the more recent work of Russo and Steen,¹² Bezdeneznykh *et al.*¹³ and a review by Finn.¹⁴

For the system under consideration here, the variation of the total energy E is

$$\delta E = \delta U_{\text{sur}} + \delta U_G + \delta U_L = \delta U_{\text{sur}} - \delta W_G - \delta W_L + \delta Q_{L \rightarrow G} + \delta Q_{G \rightarrow L}. \tag{2}$$

Here, U_{sur} is the gas-liquid surface energy, U_G and U_L are the internal energies of the gas and the liquid, W_G and W_L represent the work done on the gas and the liquid, respectively. The net heat flux is equal to zero ($\delta Q_{L \rightarrow G} + \delta Q_{G \rightarrow L} = 0$). Furthermore, $\delta W_L = P_L \delta \nu_L = 0$ because the liquid is assumed to be incompressible, i.e., $\delta \nu_L = 0$. As a result, we are left with

$$\delta E = \delta U_{\text{sur}} - \delta W_G = \delta U_{\text{sur}} - \sum_{i=1}^m P_i \delta \nu_i, \tag{3}$$

where P_i and ν_i are, respectively, the gas pressure and volume of the i th gaseous region. Taking into account the equilibrium conditions that follow from the equality $\delta E = 0$ and restriction (1), the second variation of the total energy can be presented (see Ref. 3) in the form

$$\delta^2 E = \delta^2 U_{\text{sur}} - \sum_{i=1}^m \delta P_i \delta \nu_i. \tag{4}$$

For a polytropic process in a real gas, $P_i \nu_i^n = \text{const}$, hence

$$\delta P_i = - \frac{n P_i}{\nu_i} \delta \nu_i. \tag{5}$$

From (4) and (5) we then obtain

$$\delta^2 E = \delta^2 U_{\text{sur}} + n \sum_{i=1}^m \frac{P_i}{\nu_i} (\delta \nu_i)^2, \tag{6}$$

where the positive second term on the right-hand side determines the stabilization effect mentioned above. The exponent n lies in the interval $1 \leq n \leq \eta$, where η is the adiabatic exponent of the gas. Thus, we have the lowest stabilization for isothermal ($n = 1$) process, and the highest stabilization for adiabatic process.

Based on principle of minimum potential energy, the method of calculating the stability of a disconnected free surface was described in Refs. 15 and 16, and later in Ref. 3 (pp. 207–218). (The adiabatic process considered therein can be generalized with no essential changes to the polytropic process.) For the polytropic process under consideration, the internal energy changes in the liquid and the gas influence the stability through mediation of the parameters

$$T_i = n P_i / \sigma \nu_i \quad (i = 1, 2, \dots, m), \tag{7}$$

where σ is the surface tension on the free surface.

When a free surface bounding the liquid portion consists of $m > 1$ axisymmetric connectivity components Γ_i ($i = 1, 2, \dots, m$), the problem of stability to axisymmetric perturbations is reduced to the determination of the sign of the smallest eigenvalue $\lambda^{(1)}$ of the following spectral problem:

$$-\varphi_i'' - \frac{r_i'}{r_i} \varphi_i' + a_i \varphi_i + \xi_i + \mu = \lambda \varphi_i \tag{8}$$

$$(s_{i1} \leq s_i \leq s_{i2}; \quad i = 1, \dots, m),$$

$$l_1 \varphi_i \equiv -\varphi_i'(s_{i1}) + \chi_{i1} \varphi_i(s_{i1}) = 0, \tag{9}$$

$$l_2 \varphi_i \equiv \varphi_i'(s_{i2}) + \chi_{i2} \varphi_i(s_{i2}) = 0,$$

$$l_3 \varphi_i \equiv \xi_i - 2\pi T_i \int_{s_{i1}}^{s_{i2}} r_i \varphi_i ds_i = 0, \tag{10}$$

$$l_4 \varphi_i \equiv \sum_{i=1}^m \int_{s_{i1}}^{s_{i2}} r_i \varphi_i ds_i = 0. \tag{11}$$

The equilibrium is stable to axisymmetric perturbations if $\lambda^{(1)} > 0$, and is unstable if $\lambda^{(1)} < 0$. The above equations take into account that the equilibrium surface Γ_i is symmetric about the axis z_i and has a parametric representation $r_i = r_i(s_i)$, $z_i = z_i(s_i)$. Here, r_i , θ_i , and z_i are the cylindrical coordinates, and s_i is the arc length of any axial section $\theta_i = \text{const}$ of Γ_i . This section determines the profile of the surface Γ_i with end points $s_i = s_{i1}$ and $s_i = s_{i2}$ of the equilibrium profile ($s_{i1} \leq s_i \leq s_{i2}$). The function $\varphi_i = \varphi_i(s_i)$ is a component of an axisymmetric perturbation of Γ_i normal to Γ_i , and μ is an unknown constant. The function $a_i(s_i)$ for the case of zero gravity, and the boundary parameters χ_{i1} and χ_{i2} have the following form:

$$a_i(s_i) = -(z_i'/r_i)^2 - k^2(s_i), \tag{12}$$

$$\chi_{i1} = \frac{k(s_{i1}) \cos \alpha - \bar{k}_{i1}}{\sin \alpha}, \quad \chi_{i2} = \frac{k(s_{i2}) \cos \alpha - \bar{k}_{i2}}{\sin \alpha}.$$

Here, $k(s_i)$ is the curvature of the equilibrium profile, and \bar{k}_{i1} and \bar{k}_{i2} are the curvatures of an axial section of the container at points of contact with the equilibrium profile. The boundary conditions (9) are written for a doubly connected surface Γ_i (both ends $s_i = s_{i1}$ and $s_i = s_{i2}$ of the equilibrium profile lie on a container wall). If the i th surface is simply connected, one of the ends lies on the symmetry axis z_i . The corresponding i th boundary condition in (9) must be replaced by the requirement that the solution φ_i be bounded at this end. If Γ_i does not contact the container wall (a closed sphere discussed above), φ_i must be bounded at both ends that lie on the axis z_i . Finally, Eq. (10) defines the constant ξ_i , and (11) serves as an analogy to Eq. (1) and enforces the constraint that the liquid volume be conserved.

In addition to the problem defined by Eqs. (8)–(11) we consider an auxiliary problem with modified boundary conditions corresponding to pinned contact lines. For doubly connected and simply connected Γ_i , these conditions have the form

$$\varphi_i(s_{i1}) = \varphi_i(s_{i2}) = 0 \quad \text{and} \quad \varphi_i'(s_{i2}) = 0, \tag{13}$$

respectively. The boundary condition (13) arise if all values $\chi_{i1} = \chi_{i2} = \infty$. Since the eigenvalues of problems (8)–(11) increase with each χ_{i1} and χ_{i2} , then $\lambda^{(1)}$ is less than the smallest eigenvalue $\vartheta^{(1)}$ of the auxiliary problems (8), (10), (11), and (13). Thus, the inequality $\vartheta^{(1)} > 0$ is the necessary condition for stability.

Henceforth we will consider a cylindrical container and two connectivity components with profiles that represent arcs of circles. Then $m=2$, and

$$r_i(s_i) = R_i \sin \frac{s_i}{R_i}, \quad z_i(s_i) = R_i \left(q_i - \cos \frac{s_i}{R_i} \right), \tag{14}$$

$$a_i = - \frac{2}{R_i^2},$$

$$\chi_{i1} = \chi_{i2} = \pm \frac{1}{R_i} \cot \alpha, \tag{15}$$

where R_i is the radius of the spherical surface Γ_i , and q_i is a constant. In (15) one should use the “+” (“-”) sign if Γ_i is convex toward the liquid (the gas).

Taking the relations given by (14) into consideration, we get the following solution of Eq. (8) for $\lambda=0$:

$$\varphi_i = c_{i1} u_{i1}(s_i) + c_{i2} u_{i2}(s_i) + (\xi_i + \mu) u_{i3}, \tag{16}$$

$$u_{i1} = \cos \frac{s_i}{R_i}, \quad u_{i2} = 1 + \cos \left(\frac{s_i}{R_i} \right) \ln \left(\tan \frac{s_i}{2R_i} \right), \tag{17}$$

$$u_{i3} = \frac{R_i^2}{2}.$$

Here, c_{i1} and c_{i2} are constants, and u_{i1} and u_{i2} are the solutions of corresponding homogeneous equation, and are, respectively, bounded and unbounded at the point $s_i=0$ that belongs to the axis z_i . Substituting (16) and (17) into Eqs. (9)–(11), we obtain the following condition for the existence of nontrivial solutions φ_i for $\lambda=0$:

$$D_0 \equiv \det(l_j u_{in}) = 0 \quad (j=1, \dots, 4; \quad i=1, 2; \quad n=1, 2, 3). \tag{18}$$

Equation (18) determines values of χ_{i1} and χ_{i2} for which some eigenvalue of problems (8)–(11) is equal to zero. The smallest eigenvalue $\lambda^{(1)}$ may be equal to zero only if $\vartheta^{(1)} > 0$. To obtain the stability conditions, we first need to verify that $\vartheta^{(1)} > 0$ and then we seek the branch of solutions for which $\lambda^{(1)} = 0$. This branch of solutions of Eq. (18) will have minimum values of χ_{i1} and χ_{i2} that are greater than the maximum values for all other branches. For this branch, the values of χ_{i1} and χ_{i2} are critical and we denote them as χ_{i1}^* and χ_{i2}^* , respectively. The equilibrium is stable (unstable) to axisymmetric perturbations if the values of χ_{i1} and χ_{i2} determined by (15) lie inside (outside) a stability region that is bounded by a critical branch. This is the region where one of the values, say χ_{i2} , is greater than the corresponding critical value χ_{i2}^* provided that the other values of χ_{i1} and χ_{i2} are equal to their critical values.

For each simply connected surface Γ_i , we have only two functions u_{i1} and u_{i3} instead of three functions (17), and the order of the determinant D_0 is decreased by one. For each Γ_i that is a closed sphere, we consider only the function u_{i3} (the function u_{i1} corresponds to an arbitrary transition of a spherical bubble that can be neglected), and the order of D_0 is decreased by two.

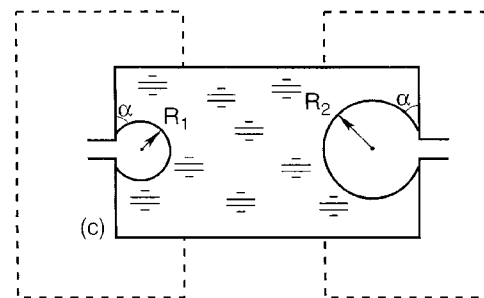
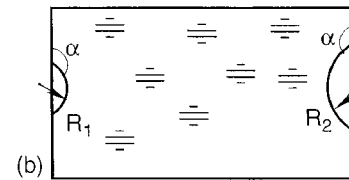
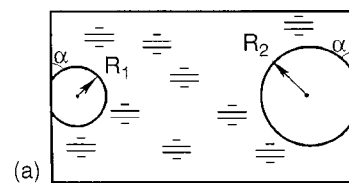


FIG. 2. Geometry of the configuration II for: (a) wetting liquid ($\alpha < 90^\circ$); (b) nonwetting liquid ($\alpha > 90^\circ$); (c) the open system corresponding to $\alpha < 90^\circ$.

Finally, note that all systems with spherical connectivity components are stable to nonaxisymmetric perturbations. Therefore, for these systems, we need only examine stability with respect to axisymmetric perturbations.

III. RESULTS

(1) *Configuration I* is the simplest configuration consisting of two isolated spherical bubbles that lie within the bulk liquid [Fig. 1(a)]. Introducing two dimensionless parameters

$$A = \frac{n P_L R_2}{\sigma}, \quad K = \frac{R_1}{R_2}, \tag{19}$$

the stability condition can be expressed as follows:

$$A > A^* = - \frac{2(3n-1)(1+K^4)}{3K(1+K^3)}. \tag{20}$$

Here, P_L is the pressure in the liquid, and $K \leq 1$. It follows that $A^* < 0$ inasmuch as $n \geq 1$, and $A > 0$ because P_L is positive in equilibrium.¹⁷ The configuration is, therefore, stable.

(2) *Configuration II* consists of two spherical bubbles that are in contact with a common bottom or with opposite bottoms [Figs. 2(a) and 2(b)]. We found that the stability condition coincides with that for configuration I given by (20). At first glance it might seem strange that the stability is independent of the wetting angle. To gain better insight into this we compared the second variation $\delta^2 E$ of the total energies for the configurations I and II in the neighborhood of their equilibrium points. The expression for $\delta^2 E$ has the following form:³

$$\begin{aligned} \delta^2 E = & 2\pi\sigma \sum_{i=1}^2 \int_0^{s_{i2}} \left(a_i \varphi_i - \varphi_i'' - \frac{r_i'}{r_i} \varphi_i' \right) r_i \varphi_i ds_i \\ & + 2\pi\sigma \sum_{i=1}^2 r(s_{i2}) \varphi_i(s_{i2}) \left[\varphi_i'(s_{i2}) + \chi_{i2} \varphi_i(s_{i2}) \right] \\ & + 4\pi^2 \sigma \sum_{i=1}^2 T_i \left(\int_0^{s_{i2}} r_i \varphi_i ds_i \right)^2. \end{aligned} \tag{21}$$

For the system *I*, the arclength $s_{i2} = \pi R_i$ and $r(s_{i2}) = 0$. Hence, the terms in the second summation on the right-hand side of (21) are zero.

The most dangerous situation for configuration *I* corresponds to the case when perturbed surfaces $\bar{\Gamma}_i$ are closed spheres concentric to the equilibrium spheres Γ_i . For this case, the perturbations $\varphi_i = c_i$ ($i = 1, 2$) are the constants related by $c_2 = -K^2 c_1$ that satisfy the liquid volume conservation constraint. From Eq. (21), we then obtain an expression for the smallest value of $\delta^2 E$ for the system *I*:

$$\delta^2 E_I = 4\pi\sigma c_1^2 \left[-2(1 + K^4) + \frac{3n}{\sigma} (P_1 R_1 + P_2 R_2 K^4) \right]. \tag{22}$$

For configuration *II*, the most dangerous case corresponds to perturbed surfaces $\bar{\Gamma}_i$ ($i = 1, 2$) that represent spherical caps with their centers displaced along the axes z_i for distances ϵ_i with respect to the centers of equilibrium caps Γ_i . The perturbed surfaces, like the surfaces Γ_i , meet the flat endwalls at angle α , so that $\epsilon_i = -(\bar{R}_i - R_i) \cos \alpha$, where \bar{R}_i is the radius of the perturbed sphere $\bar{\Gamma}_i$. Then

$$\varphi_i = -\epsilon_i \left(\frac{1}{\cos \alpha} + \cos \frac{s_i}{R_i} \right) \quad (i = 1, 2), \quad \epsilon_2 = -K^2 \epsilon_1, \tag{23}$$

and the smallest value of $\delta^2 E$ for configuration *II* is

$$\begin{aligned} \delta^2 E_{II} = & \pi\sigma \epsilon_1^2 \frac{(2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha)}{\cos^2 \alpha} \left[-2(1 + K^4) \right. \\ & \left. + \frac{3n}{\sigma} (P_1 R_1 + P_2 R_2 K^4) \right]. \end{aligned} \tag{24}$$

The expressions (22) and (24) are identical to within a positive factor that depends on α . Thus, the stability conditions for configurations *I* and *II* are also coincident. Notice that, for stability,

$$-2(1 + K^4) + \frac{3n}{\sigma} (P_1 R_1 + P_2 R_2 K^4) > 0. \tag{25}$$

This is equivalent to the stability condition (20).

The first term in (25) is related to surface forces and is negative. This means that the system is always unstable if the work done on the gas is negligible. This is true for the ‘‘open’’ system, in which each gas cavity is in contact with a large gas reservoir such that ν_1 and ν_2 are both large [Fig. 2(c)]. The loss of stability of the open system subject to perturbations of the form (23) causes one of the gas cavities to spread along the endwall and results in a decrease in contact area between the other gas cavity and the endwall on

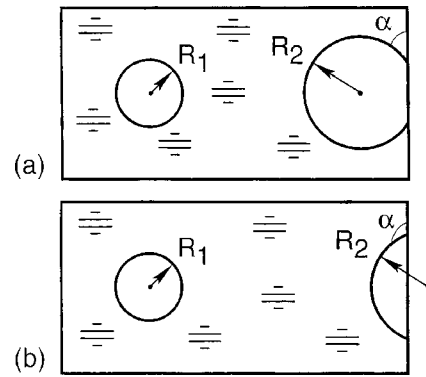


FIG. 3. Configuration *III*: (a) wetting liquid; (b) nonwetting liquid.

which it resides. This may bring the system to a new stable equilibrium state with a different geometry. For example, the free surface of the smaller bubble may be pinned to the edge of a hole in the endwall and there will be no constraint on the angle α for this bubble or, alternatively, the liquid may simply drain through the hole.

The stability condition (25) can also be expressed in terms of T_1 and T_2 . For the configuration *II*, it has the form

$$T_1 + T_2 > \frac{2(1 + K^4)}{\pi R_1^4 (1 + \cos \alpha)^2 (2 - \cos \alpha)}. \tag{26}$$

Before concluding this section, we consider the type of ‘‘partly’’ open systems that correspond to the configuration *II*. Here, one of the gas cavities is connected to a large gas reservoir. For the case $T_2 = 0$ and $T_1 > 0$, when the cavity bounded by Γ_2 ($R_2 > R_1$) is connected to the reservoir, the stability condition

$$n(2 + P_L R_1 / \sigma) > 2(1 + K^4) / 3, \tag{27}$$

is always fulfilled for $P_L > 0$. When the smaller radius cavity, Γ_1 , is connected to the reservoir, then $T_1 = 0$ and $T_2 > 0$, and the stability condition

$$n(2 + P_L R_2 / \sigma) > 2(1 + K^4) / (3K^4), \tag{28}$$

is fulfilled for relatively large K and fails at small K .

Thus, whereas the configuration *II* in a closed cylinder is stable for positive P_L , the system shown in Fig. 2(c) is always unstable. The related system with $T_2 = 0$ and $T_1 > 0$ is stable, while the stability of the system with $T_1 = 0$ and $T_2 > 0$ depends on value of K .

(3) *Configuration III*. Although in this configuration a free surface consists of a closed sphere and a spherical cap in contact with a planar bottom (Fig. 3), its stability condition is distinct from those of configurations *I* and *II*. The parameters (19) are employed again and the subscript 1 is taken to refer to the closed sphere. Now the value of K may be less than, equal to or greater than one. The stability condition has the form

$$\begin{aligned} A > A^* \\ = & - \frac{2(3n - 1)(2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha + 4K^4)}{3K(2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha + 4K^3)}. \end{aligned} \tag{29}$$

Since A is always positive, this configuration is stable.

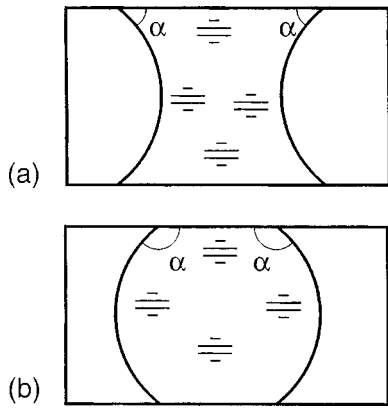


FIG. 4. Configuration IV: (a) $\alpha < 90^\circ$; (b) $\alpha > 90^\circ$.

It is useful to consider the case of the partly open system where the bubble on the endwall is connected to a large volume reservoir ($T_2 = 0$). In this case, A^* takes the simpler form

$$A^* = \frac{2}{3K} \left(1 - 3n + \frac{4K^4}{2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha} \right). \quad (30)$$

This system is less stable than for a closed cylinder. In particular, it is always unstable if α is close to π .

(4) *Configuration IV*. In the sketches shown in Fig. 4, the cross sections of these configurations have a liquid bridge type appearance. However, their stability conditions are distinctly different from those of liquid bridges between flat plates (see Ref. 3, pp. 154–156, and Refs. 18–23). In the configuration under consideration the free surface consists of two spherical caps that are in contact with a cylindrical wall (Fig. 4). The cap radii are equal because the wetting angles are the same ($R_1 = R_2 = R_0 / |\cos \alpha|$, where R_0 is the cylinder's radius). This configuration is always stable. To prove it we first considered the corresponding open system in which both domains occupied by a gas are in contact either with an infinite space ($\nu_1 = \nu_2 = \infty$) or with themselves (the latter case is possible since $k(s_1) = k(s_2)$). Such a system is neutrally stable ($\lambda^{(1)} = 0$). Clearly, the open system is in a position of indifferent equilibrium since the liquid configuration may be rigidly translated along the axis of symmetry without variation in the system energy. Such displacements are the most dangerous perturbations. In a closed cylinder, the configuration is stabilized ($\lambda^{(1)} > 0$) by internal energy of the system. The same is true for a partly open cylinder when only one of gas cavities is in contact with an infinite space.

(5) *Configuration V*. Let us assume that Γ_1 relates to a closed sphere and Γ_2 to a spherical cap in contact with a lateral wall (Fig. 5). For this case we found that the most dangerous perturbations correspond to $\varphi_1 = c_1$ and $\varphi_2 = c_2 \cos(s_2/R_2)$. The function $\varphi_2(s_2)$ describes the perturbation that shifts Γ_2 along the cylinder's axis. The stability condition for this configuration can be presented in one of two equivalent forms

$$T_1 + T_2 > \frac{1}{2\pi R_1^4} \quad \text{or} \quad 3KA + 2(3n - 1) + 4\pi R_1^4 \frac{nP_2}{\sigma \nu_2} > 0. \quad (31)$$

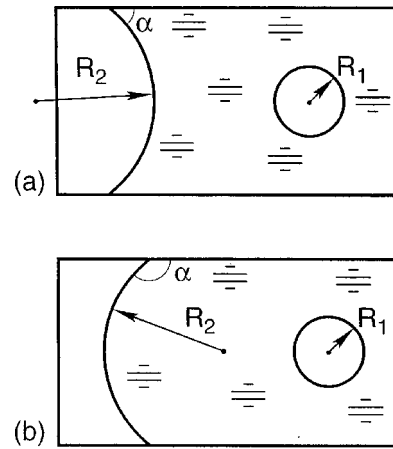


FIG. 5. Configuration V: (a) wetting liquid; (b) nonwetting liquid.

Again, even in the limiting case of $\nu_2 = \infty$, the system is stable for $P_L > 0$.

(6) *Configuration VI*. Here we consider the case when Γ_1 is a spherical cap in contact with one of the flat endwalls, and Γ_2 is in contact with the cylindrical wall (Fig. 6). For this system, the combination of perturbations $\varphi_1 = \epsilon_1 ([1/\cos \alpha] - [\cos(s_1/R_1)])$ and $\varphi_2 = c_2 \cos(s_2/R_2)$ is the most dangerous. The stability condition assumes the form

$$T_1 + T_2 > \frac{2}{\pi R_1^4 (2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha)}. \quad (32)$$

If the system is open ($\nu_1 = \nu_2 = \infty$), then stability is impossible. However, for a closed cylindrical container, we use (32) to obtain

$$3KA + 2(3n - 1) + \pi R_1^4 (2 + 2 \cos \alpha + \sin^2 \alpha \cos \alpha) \frac{nP_2}{\sigma \nu_2} > 0, \quad (33)$$

and then reach the conclusion that at least for $P_L > 0$ the system is stable. The related partly open system with $T_2 = 0$ ($\nu_2 = \infty$) is also stable. However, for the partly open system with $T_1 = 0$, the stability depends on values of the parameters. In particular, such systems are unstable for large ν_2 or for the wetting angle close to 180° .

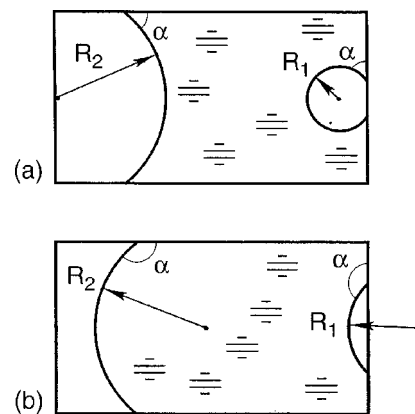


FIG. 6. Configuration VI: (a) wetting liquid; (b) nonwetting liquid.

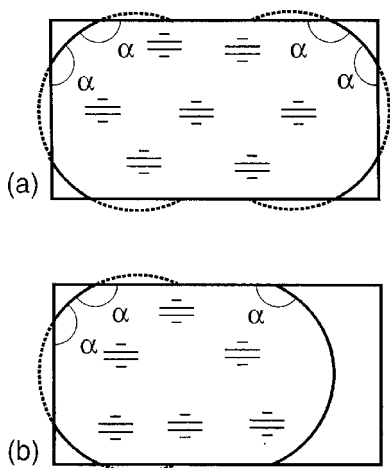


FIG. 7. Configurations (a) VII and (b) VIII, for $135^\circ < \alpha < 180^\circ$.

(7) *Configurations VII and VIII.* The remaining configurations contain a doubly connected surface that conforms to a portion of a sphere with one contact line on the lateral wall and the another on the flat endwall. Such a surface can exist only for $135^\circ < \alpha < 180^\circ$. When $\alpha \rightarrow +135^\circ$, all the configurations in a closed cylinder are stable due to the stabilizing effect of the internal energy. In this case, the volume of the gas cavity tends to zero and, hence, the associated parameter T_i tends to infinity.

In configuration VII, both connectivity components conform to the surface described above [Fig. 7(a)]. In configuration VIII, the second connectivity component is a spherical cap in contact with the cylindrical wall [Fig. 7(b)]. In these cases, the algebra necessary to obtain χ_{i1}^* and χ_{i2}^* from Eq. (18) is too cumbersome to yield a closed form expression. Instead, the related stability problems were solved using MATHEMATICA.²⁴

We have found that both these configurations are always stable even when the system's internal energy is neglected. Indeed, for configuration VII, $R_1 = R_2 = R$ and $\chi_{11} = \chi_{12} = \chi_{21} = \chi_{22} = \chi = -(1/R)\cot \alpha$. We first verified that for the related open system $v^{(1)} > 0$. Then, for this system, we set $\xi_i = 0$ in (16), eliminated the condition (10) and obtained a quartic equation in χ from Eq. (18). The critical value χ^* for the open system is the largest real root of this equation. Cal-

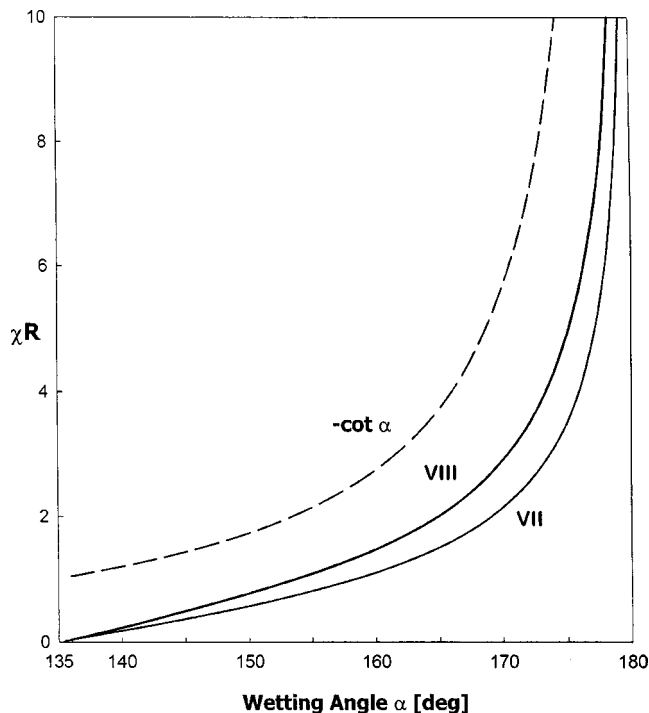


FIG. 8. Product χ^*R of critical boundary parameter, χ^* , times the spherical surface radius, R , as a function of the wetting angle, α , for open systems corresponding to configurations VII and VIII (solid lines). Dash line relates to the product $\chi R (= -\cot \alpha)$ of an actual value χ and R .

culations show that the actual value $\chi = -(1/R)\cot \alpha$ is greater than χ^* (see Fig. 8) and, hence, the open system is stable.

For the open system related to the configuration VIII, Eq. (18) yields a cubic equation in χ . For this case we also found that the actual value of χ exceeds the critical one (see Fig. 8).

(8) *Configuration IX* is bounded by a portion of a sphere, Γ_2 , and by the surface Γ_1 of a bubble inside the liquid (Fig. 9). The stability condition assumes the form

$$T > T^* = f_1(\alpha, Q), \tag{34}$$

where

$$T = \pi R_0^4 (T_1 + T_2), \tag{35}$$

$$f_1(\alpha, Q) = \frac{1}{2Q^4} \frac{2 \cos^3 \alpha}{3 \sin^2 \alpha - \sin \alpha \cos \alpha + 2 \tan \alpha + \cos^3 \alpha \ln[\tan(\alpha/2 - \pi/4)\tan(\alpha/2)]}, \tag{36}$$

and $Q = R_1/R_0$. For any given Q ($0 < Q < 1$), the function $f_1(\alpha, Q)$ increases monotonically from $-\infty$ to $0.5 Q^{-4}$ as α varies from 135° to 180° . It can be easily verified that the value of T is always greater than $0.5 Q^{-4}$ even for the case of a partly open system with $T_2 = 0$. (Such a system can be realized if $P_2 = 0$ or if $v_2 = \infty$, when the gas cavity bounded

by Γ_2 is assumed to be in contact with a large gas reservoir, for example, through a hole in the endwall). In this case,

$$P_1 \geq 2\sigma(R_1^{-1} + R_2^{-1}), \tag{37}$$

for $P_2 \geq 0$, so that

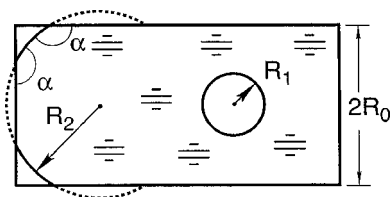


FIG. 9. Sketch of the configuration IX ($135^\circ < \alpha < 180^\circ$).

$$T = \pi R_0^4 n P_1 / \sigma \nu_1 \geq 1.5n(1 - Q \cos \alpha)Q^{-4}, \quad (38)$$

and $T > 0.5 Q^{-4}$ since $1 \leq n \leq \eta$. Thus, the partly open system is always stable due to the stabilizing effect of the gas inside the ‘‘isolated’’ bubble. The closed system is even more stable.

(9) Configuration X has a free surface consisting of the portion of a sphere, Γ_2 , and a spherical cap, Γ_1 , in contact with the bottom (Fig. 10). The configuration’s geometry is conveniently characterized by the ratio of the radius of the contact line on the endwall to the cylinder’s radius, $G = -K \tan \alpha$ ($0 < G < 1$). First we found that the corresponding open system is unstable. Then we calculated the critical values of the parameter T [see Eq. (35)]. In contrast to configuration IX, the critical values T^* cannot be presented in analytical form. For the cases of $G = 0.1, 0.5$, and 0.9 , computed values are shown in Fig. 11. Values of T^* like those for the previous configuration IX are independent of n . For any fixed G , the T^* values increase monotonically from $-\infty$ to a limiting value $L = L(G)$ as the wetting angle varies from 135° to 180° .

By estimating the actual value of T for partly open system corresponding to $T_2 = 0$, we were able to ascertain that such a system is stable. Indeed, using (37), we obtain

$$T = \frac{\pi R_0^4 n P_1}{\sigma \nu_1} \geq f_2(\alpha, G, n), \quad (39)$$

where

$$f_2(\alpha, G, n) = \frac{6n \sin^3 \alpha (\sin \alpha - G \cos \alpha)}{G^4 (1 + \cos \alpha)^2 (2 - \cos \alpha)}. \quad (40)$$

For given G and n , the function f_2 increases with α from its minimum value $M = M(G, n)$ at $\alpha = 135^\circ$ and tends to infinity as α approaches to 180° . Note that M is always greater than the L for the same G . For example, for $G = 0.1, 0.5$, and 0.9 (illustrated in Fig. 11), $M = 71\,049, 155$, and 18.7 , respectively, for the most dangerous case of $n = 1$. Thus, T

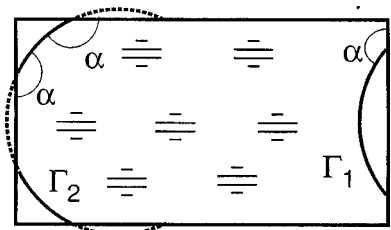


FIG. 10. Configuration X ($135^\circ < \alpha < 180^\circ$).

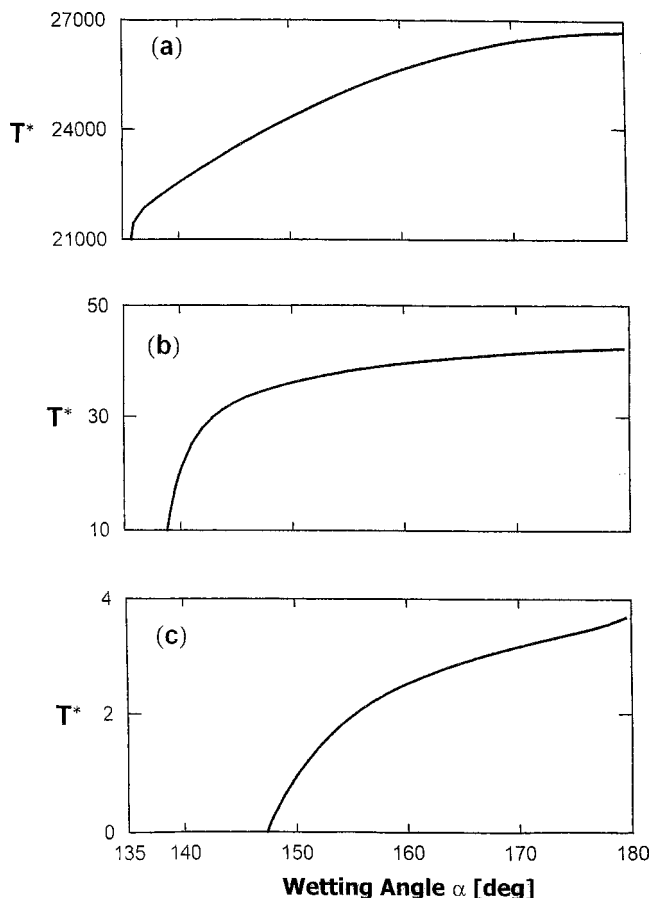


FIG. 11. Dependence of the critical value of T on the wetting angle for the configuration X with: (a) $G = 0.1$; (b) $G = 0.5$; (c) $G = 0.9$.

$> T^*$ for the partly open system with $T_2 = 0$. This means that both partly open system with $T_2 = 0$ and the closed configuration X are stable.

In contrast, the related partly open system with $T_1 = 0$ is always stable only for α values from some neighborhood of 135° . In this interval, T^* is negative. The greater the value of G , the wider the interval. If T^* is positive, the stability depends on the value of T_2 , i.e., on the pressure P_2 . The partly open system under consideration is stable at high pressure, and is unstable at low P_2 . It follows from Fig. 11 that for given values of α and T_2 the partly open system with $T_1 = 0$ is stable at G close to unity, and is unstable at small G .

IV. CONCLUSIONS

Generally a disconnected surface has the same or lower level of stability compared with the worst stability conditions for cases when each Γ_i is considered as the sole connectivity component. Using simple problems we have demonstrated that an important effect of disconnectedness on the stability of the free surface of a liquid in a partly filled container is the stabilizing effect of the system’s internal energy. Indeed, for the cases we considered, the sole connectivity components have identical ‘‘levels’’ of stability—that is, they are stable. For the ten different combinations of various connectivity

components considered here, we concluded that the configuration is either stable or its stability depends on the contribution from the internal energy.

Note that only the configurations *IV*, *VII*, and *VIII* necessarily have a common axis of symmetry that coincides with the cylinder's axis. The others do not need to be symmetric about this axis. Furthermore, the stability results obtained for configurations *I–III* are not restricted to a cylindrical container. Configuration *I* relates to any container, while configurations *II* and *III* may arise in any container with one planar wall. Thus, application of the results obtained for systems *I–III* is not limited to cylindrical containers.

We have analyzed only simple disconnected surfaces with $m=2$ when each connectivity component has a spherical shape. It should be emphasized that the spherical shape of axisymmetric connectivity components in configurations *I–VI* is unique. From this point of view the related results are complete. In contrast, for configurations *VII–X*, a connectivity component that conforms to a portion of a sphere is only one of a number of possible axisymmetric doubly connected surfaces (spherical, catenoidal, nodoidal and unduloidal). For $m=1$, a spherical doubly connected surface is always stable, while other surfaces may be stable or unstable depending on the values of α and the relative volume of the gaseous region.^{7,8} To complete the investigation of the configurations like that *VII–X* other types of doubly connected surfaces must be considered. Then, as was done for $m=1$ in Refs. 7 and 8, the stability region can be obtained for related configurations with $m=2$.

Furthermore, we did not analyze any configuration where one of connectivity components is an axisymmetric doubly connected surface with both contact lines on the lateral wall. For $m=1$ and consequently for $m=2$, stability is possible only if this is an unduloidal surface.^{7,8} Thus, five more configurations remain to be examined to gain greater insight into possible stable configurations of a weightless liquid in a cylindrical container.

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